Asymptotical unboundedness of the Heesch number in $\mathbb{E}^d$ for $d \to \infty$

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Abstract

We solve d-dimensional Heesch's problem in the asymptotic sense. Namely, we show that, if we let $d \to \infty$, then there is no uniform upper bound on the set of all possible finite Heesch numbers in the space $\mathbb{E}^d$; in other words, given any nonnegative integer n, we can find a dimension d (depending on n) in which there exists a hypersolid whose Heesch number is finite and greater than n.

Mathematics Subject Classification (2020): 52C22, 05B45, 51M20

Keywords: Heesch number, Heesch's problem, d-dimensional Euclidean space, tessellation, tiling

1 Introduction

Problems on *tessellations* (or *tilings*) are a very active research direction within the branch of combinatorial geometry. Recall that a tessellation of the Euclidean plane $\mathbb{E}^2$ is a set $\mathscr{T}$ such that each $T \in \mathscr{T}$ is a closed topological disc, every two different $T', T'' \in \mathscr{T}$ have disjoint interiors, and $\bigcup \mathscr{T} = \mathbb{E}^2$. The elements of $\mathscr{T}$ are called *tiles*. The book by Grünbaum and Shephard [2] is a very comprehensive treatment of the theoretical foundations of tilings, and it is a very valuable source of information even still today.

As far as this article is concerned, all the tessellations will consist of mutually congruent tiles (the so-called *monohedral* tessellations); if there exists a tessellation such that all the tiles are congruent to a given figure, we shall say that such figure *tessellates* the plane.

Given two figures, each of which tessellates the plane, one of them can be deemed "harder to tessellate by" than the other. The so-called *isohedral number* of a figure is a positive integer that presents one such measure (the larger isohedral number is, the figure is "more complex"). Answering the second part of Hilbert's 18^{th} problem, in 1928 Reinhardt [8] constructed a (three-dimensional) polyhedron whose isohedral number is 2 (which was the first example with an isohedral number greater than 1). Grünbaum and Shephard share the opinion [2, Section 9.6] that the Hilbert's problem was formulated in three dimensions because Hilbert believed that in two dimensions such a figure does not exist. However, in 1935 Heesch [3] constructed a two-dimensional example. Today it is known, as shown by Socolar [9], that in $\mathbb{E}^d$ for $d \geqslant 3$ there exists for each k a (hyper)solid whose isohedral number is k, but in two dimensions this is still an open problem (the current record-holder is a figure whose isohedral number is 10, constructed by Myers [7]).

The same Heesch introduced [4] what today is known as the *Heesch number*, which ranks figures that do *not* tessellate the plane by their ability to "advance" toward a tessellation; the Heesch number is a nonnegative integer such that, the larger it is, the given figure can advance "further" toward a tessellation (if a figure tessellates the plane, it is convenient to define the Heesch number of that figure to be infinite). Speaking somewhat informally (a formal definition will be given later), we define the Heesch number of a given figure T to be the maximal nonnegative integer n such that T can be completely surrounded by congruent copies of itself n times in total. See Figure 1 for an example: there we have a convex pentagon (that can be obtained by gluing together a square, an equilateral triangle, and a 30°-60°-90° triangle) that is surrounded once. Note that there are more possible ways to surround this figure, two of which are shown, which differ even by the number of copies used for surrounding (though this quantity is of no significance for the Heesch number). We would like to add that, whenever this pentagon is shown in the literature, practically always it is shown surrounded by either 7 or 8 copies, but as can be seen here, it can also be surrounded by 6 or 9 copies. We have calculated (with a computer help) that there exist exactly 2740 possible noncogruent first coronas around this pentagon (where coronas that differ only by a translation of a "loose" tile, such as the topmost tile in Figure 1 left, are considered to be in the same congruence class), of which there are only 122 that consist of 6 copies, and even less, only 64, that consist of 9 copies (in comparison to a total of 1234 those that consist of 7 copies and 1320 those that consist of 8 copies). However, it can be shown that, no matter how we surround the observed figure, the resulting shape can never be surrounded one more time by copies of the given figure; therefore, the Heesch number of the observed figure equals 1 (this example has been found by Heesch himself [4]).

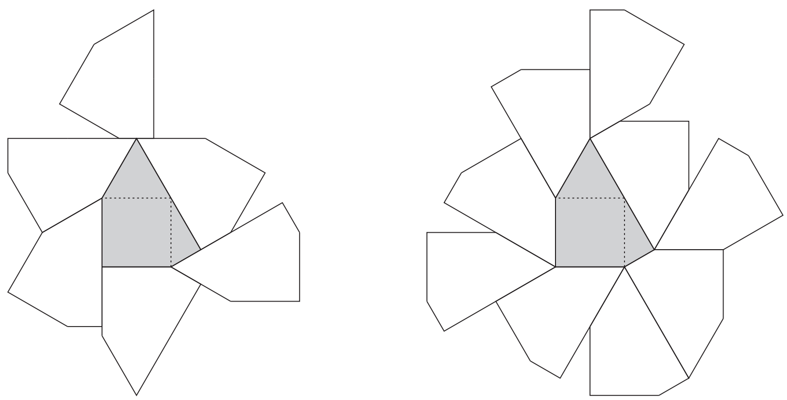

Figure 1: A figure surrounded once by its congruent copies (two ways are shown).

Probably the most important open problem concerning the Heesch number (called *Heesch's problem*) asks whether the set of all finite values that can be the Heesch number of some figure is bounded from above (in other words, whether there exists the largest possible finite Heesch number). The current record-holder is a figure whose Heesch number is 5 (actually, a family of figures, each of which has the Heesch number 5), constructed by Mann [5]. We also note that Heesch's problem has been solved in the hyperbolic plane [10], as well as in the version for sets of more figures (at least three) [1]; in both these cases, it has been shown that there is no upper bound on the set of all possible finite Heesch numbers.

So far, all the research on the Heesch number has been done pretty much exclusively within the two-dimensional space (that is, the plane). Some suggestions that it would be useful to study the problem in larger-dimensional spaces appeared in [6], but remained largely unexplored in the literature. In this article we solve d-dimensional Heesch's problem in the asymptotic sense. Namely, we show that, if we let $d \to \infty$, then there is no uniform upper bound on the set of all possible finite Heesch numbers in the space $\mathbb{E}^d$; in other words, given any nonnegative integer n, we can find a dimension d (depending on n) in which there exists a hypersolid whose Heesch number is finite and greater than n.

2 The main section

Let us first formally define the necessary notions.

Definition 1. We say that a hypersolid C (a topological d-ball) in $\mathbb{E}^d$ can be *surrounded n times* if and only if there exist finite collections $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ of isometric copies of C such that:

- every two different hypersolids from $\{C\} \cup \bigcup_{i=1}^{n} \mathscr{C}_i$ have disjoint interiors;
- for each i, $1 \leqslant i \leqslant n$, each hypersolid from $\mathscr{C}_i$ has a common boundary point with some hypersolid from $\mathscr{C}_{i-1}$ (where by convention, we let $\mathscr{C}_0 = \{C\}$);
- for each i, $1 \leqslant i \leqslant n$, $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j \right)$ is a closed topological d-ball such that $\bigcup \left(\bigcup_{j=0}^{i-1} \mathscr{C}_j \right)$ is completely contained in its interior.

The collection $\mathscr{C}_i$ is called the i^{th} *corona*.

Definition 2. The *Heesch number* of a given hypersolid C (a topological d-ball) in $\mathbb{E}^d$ is the maximal nonnegative integer n such that C can be surrounded n times. If such a maximum does not exist, then we define the Heesch number to be infinite.

From now on, we work in a given Euclidean space $\mathbb{E}^d$, where $d = 2^k$ for some positive integer k. We shall define a hypersolid in $\mathbb{E}^d$ as follows. We start from a unit hypercube, and mark some of its facets (which are $(d-1)$-dimensional unit hypercubes) by "bumps" and "nicks" (arranged in a particular way that will be described in a moment), where each bump matches each nick. In particular, each bump or nick can be taken to be a right hypercone whose base is an $(n-1)$-dimensional (small) hyperball placed in the center of a facet of the considered hypercube, and whose axis is orthogonal to the facet; we call *bumps*, respectively *nicks*, such hypercones erected outwards, respectively inwards (with respect to the interior of the considered hypercube).

Definition 3. A *basic hypercube* is a hypersolid obtained in the described way that has d facets with bumps and $d-1$ facets with nicks (and 1 facet not marked by either), where, additionally, all d facets with bumps intersect at one vertex of the considered hypercube.

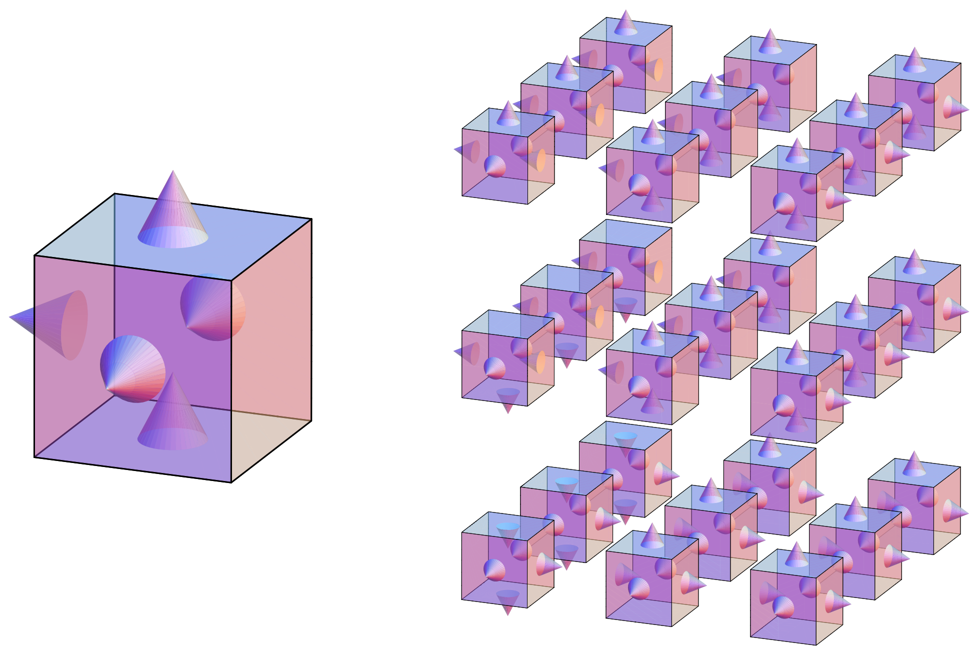

Figure 2: A 3-dimensional basic hypercube, and the same basic hypercube surrounded by 26 isometric copies of itself.

Example. In Figure 2 left we give an example of a basic hypercube in three dimensions (for the purpose of this illustration, we ignore the fact that 3 is not of the form 2^k), while on the right we show how it can be surrounded by its 26 isometric copies (note that, therefore, the Heesch number of this solid is at least 1). In Figure 3 we give an example of a basic hypercube in four dimensions. We would like to emphasize that this image is not an "artistic impression," but it has actually been drawn by first writing all the necessary equations in 4D, and then projecting first to 3D and then finally to 2D; therefore, it is as realistic as possible.

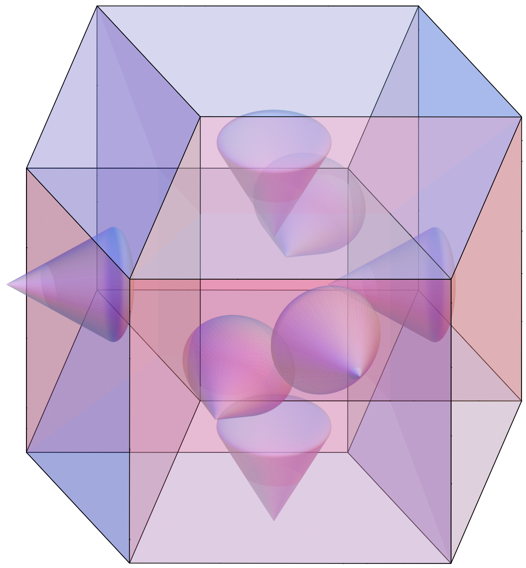

Figure 3: A 4-dimensional basic hypercube.

We first show the following lemma.

Lemma 4. *The Heesch number of a basic hypercube is at most $d - 1$.*

Proof. Suppose the contrary: a given basic hypercube can be surrounded d times. Then the resulting configuration is a hypercube of side $2d + 1$ (up to bumps and nicks), which means that it consists of $(2d+1)^d$ basic hypercubes. Note that the total number of bumps among all these basic hypercubes equals $d(2d + 1)^d$, while the total number of nicks equals $(d - 1)(2d + 1)^d$. It follows that there are at least $d(2d+1)^d - (d-1)(2d+1)^d$, that is, $(2d+1)^d$ bumps that are not matched by any nick. All such bumps have to be on the boundary of the considered hypercube; however, the hyperarea of that boundary (which is the total number of those unit facets of all the basic hypercubes that are on the boundary of the considered hypercube) equals $2d(2d + 1)^{d-1}$, which is less than $(2d+1)^d$, and thus there is not enough "room" for all the considered bumps, which is a contradiction. The lemma is thus proved. ■

We are left to prove that a basic hypercube can be surrounded $d - 1$ times by isometric copies of itself. In fact, we shall prove more: it is possible to stack $(2d)^d$ isometric basic hypercubes to form a hypercube (up to bumps and nicks) of side $2d$ (note that a hypercube of side $2d - 1$ would suffice for the assertion from the previous sentence). The stacking is (naturally) done in such a way that any two basic hypercubes that share a common facet must have the corresponding facets marked in a matching way (that is, either a bump in one hypercube and a nick in the other one, or unmarked facets in both hypercubes).

During the proof, we shall need to refer to various unit hypercubes marked by bumps and nicks in different ways. (We shall onward refer to them as *marked hypercubes.*) Therefore, let us introduce the necessary notation first. We shall consider only hypercubes whose edges are parallel to the coordinate axes. Each such hypercube with a given center will be described by a matrix

$$\begin{bmatrix} b_{1,0} & b_{1,1} \\ b_{2,0} & b_{2,1} \\ \vdots & \vdots \\ b_{d,0} & b_{d,1} \end{bmatrix} \tag{1}$$

with $b_{i,j} \in \{1, 0, -1\}$. We interpret that matrix as follows. The i^{th} row describes the two facets orthogonal to the i^{th} coordinate axis, in the order in which they are met when traveling the axis from $-\infty$ to ∞; $b_{i,j} = 1$ means that there is a bump on the corresponding facet, $b_{i,j} = -1$ means that there is a nick on the corresponding facet, while $b_{i,j} = 0$ means that the corresponding facet is not marked by neither bump nor nick.

Notice that such a matrix represents a basic hypercube if and only if the value 1 appears d times, the value -1 appears $d-1$ times, and the value 0 appears once, and additionally, the value 1 appears exactly once in each row.

Example. Consider the basic hypercube from Figure 2 left. If the x-axis goes from left to right, the y-axis from bottom to top, and the z-axis from front to back, then the corresponding matrix for that hypercube is

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In a similar manner, considering the basic hypercube from Figure 3, if the x-axis goes from left to right, the y-axis from bottom to top, and the two more axes from front to back, the corresponding matrix for that hypercube is

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

We observe that two marked unit hypercubes are isometric if and only if the corresponding matrices can be transformed one into another by a permutation of rows, or mutually swapping the two values from the same row (or any combination of these operations); indeed, the first operation represents a permutation of coordinate axes, while the second one represents switching the direction of a particular axis. We also note that any two basic hypercubes are (clearly) isometric.

Finally, we shall also need the following definition.

Definition 5. For $t \in \{1, 2, \ldots, k+1\}$ (recall that $d = 2^k$), a *sample of order* t is a marked unit hypercube whose corresponding matrix contains the value -1 exactly $2^k - 2^{t-1}$ times, all the other entries are equal to 1, and there is no row in which both entries are equal to -1.

Note that a sample of order t is not a basic hypercube, it is only a marked hypercube. Clearly, for a given t, any two samples of order t are isometric.

We now prove a series of lemmas showing some connections between basic hypercubes and samples, as well as connections among different samples. To simplify the terminology, whenever we discuss the two facets (of any hypercube) orthogonal to a given coordinate axis, then the one that is met first when traveling the axis from $-\infty$ to ∞ will be called "left," and the other one will be called "right."

Lemma 6. *We can stack* 2^d *basic hypercubes in such a way to obtain a hypercube (up to bumps and nicks) of side* 2 *that "behaves like" a sample of order* 1 *scaled by factor* 2.

Here, by "behaves like" we mean the following: for each facet of the obtained hypercube, all 2^{d-1} *unit facets that compose it are marked in the same way (that is, all bumps, all nicks, or all unmarked), and furthermore, if we replace any such ensemble of bumps/nicks by one (centered) bump/nick, we get precisely a sample of order* 1 *scaled by factor* 2. *(In effect, the obtained structure can be thought of as a sample of order* 1 *where bumps/nicks have a geometrically different shape—*2^{d-1} *hypercones instead of one—but still each* *bump matches each nick, and the structure is practically equivalent to a sample of order* 1 *in every meaningful manner.)*

Proof. We shall arrange 2^d basic hypercubes in such a way that their centers are at the coordinates $\{0,1\}^d$. (By "center" of a basic hypercube we mean the center of the underlying unit hypercube—not the geometrical centroid, which would be slightly off because of the attached hypercones.) The matrix (1) that describes the basic hypercube whose center is at $(c_1, c_2, \ldots, c_d)$ has $b_{i,0} = 1$ and $b_{i,1} = -1$ for all $i \geqslant 2$, and

$$\begin{cases} b_{1,0} = 1 \text{ and } b_{1,1} = 0, & \text{if } c_1 = 0; \\ b_{1,0} = 0 \text{ and } b_{1,1} = 1, & \text{if } c_1 = 1. \end{cases}$$

Let us first show that every two neighboring of these basic hypercubes have the common facet marked in a matching way. And indeed, the basic hypercubes centered at $(c_1, c_2, \ldots, c_d)$ and $(c'_1, c'_2, \ldots, c'_d)$ are neighbors if and only if for some i we have $c_i = 0$ and $c'_i = 1$ (or vice versa), and $c_j = c'_j$ for each j, $j \neq i$. If their corresponding matrices are B and B', we need to check whether $b_{i,1} = -b'_{i,0}$. And indeed, by our construction we have $b_{i,1} = -1 = -b'_{i,0}$ whenever $i \geqslant 2$, while for $i = 1$ we have $b_{1,1} = 0$ and $b'_{1,0} = 0$ (since $c_1 = 0$ and $c'_1 = 1$), in both cases as needed.

We are left to show that the obtained hypercube behaves like a sample of order 1. Consider the two facets of the obtained hypercube orthogonal to the i^{th} coordinate axis. The left one is composed of 2^{d-1} left facets (with respect to the same axis) of all the basic hypercubes whose center has 0 at the i^{th} coordinate; by our construction, each of them is marked by a bump. In a similar way, we see that the right facet is composed of 2^{d-1} facets each of which is marked by a nick, with an exception of the case $i = 1$, when each of them is marked by a bump.

Altogether, there are $2^{d-1} - 1$ facets of the obtained hypercube marked (solely) by nicks, and all the other facets are marked (solely) by bumps; furthermore, there is no i such that both facets orthogonal to the i^{th} coordinate axis are marked by nicks (since all the facets marked by nicks are right ones). Therefore, the obtained hypercube behaves like a sample of order 1 (scaled by factor 2), which was to be proved. ■

Lemma 7. *Let* $t \in \{1, 2, \ldots, k+1\}$. *Assume that a marked unit hypercube has center at* $(c_1, c_2, \ldots, c_d)$ *with* $c_i \in \{0, 1\}$, *and is represented by a* $d \times 2$ *matrix* B *given by:*

- $b_{i,c_i} = 1$ *and* $b_{i,1-c_i} = (-1)^{i+\sum_{u=1}^{d} c_u}$ *for* $1 \leqslant i \leqslant 2^t$*;*

- $b_{i,0} = 1$ *and* $b_{i,1} = -1$ *for* $2^t < i \leqslant 2^k$*.*

Then the considered marked hypercube is a sample of order t*.*

Proof. Clearly, the only values in B are 1 and -1, and there is no row in B in which both entries are equal to -1. Therefore, we are left to prove that -1 appears exactly $2^k - 2^{t-1}$ times in B. Since $\sum_{u=1}^{d} c_u$ is constant, we have that, for $1 \leqslant i \leqslant 2^t$, exactly half of the values $(-1)^{i+\sum_{u=1}^{d} c_u}$ equal -1. Therefore, -1 appears exactly 2^{t-1} times within the first 2^t rows. Further, each of the next $2^k - 2^t$ rows contains the value -1 exactly once. That makes a total of $2^{t-1} + 2^k - 2^t$, that is, $2^k - 2^{t-1}$ times the value -1 appears in B, which was to be proved. ■

Lemma 8. *Let* $t \in \{1, 2, \ldots, k\}$*. We can stack* 2^d *samples of order* t *in such a way to obtain a hypercube (up to bumps and nicks) of side* 2 *that behaves like a sample of order* $t + 1$ *scaled by factor* 2*.*

Proof. We shall arrange 2^d samples of order t in such a way that their centers are at the coordinates $\{0, 1\}^d$. The sample centered at $(c_1, c_2, \ldots, c_d)$ will be precisely the one defined in the formulation of Lemma 7.

Let us first show that any two neighboring samples match. Say that they are centered at $(c_1, c_2, \ldots, c_d)$ and $(c'_1, c'_2, \ldots, c'_d)$, and represented by matrices B and B'. Let i be the (only) coordinate such that (w.l.o.g.) $c_i = 0$ and $c'_i = 1$. We then have to show that $b_{i,1}$ and $b'_{i,0}$ have opposite signs. If $2^t < i \leqslant 2^k$, we immediately have $b_{i,1} = -1 = -b'_{i,0}$. Assume now $1 \leqslant i \leqslant 2^t$. We then have $b_{i,1} = b_{i,1-c_i} = (-1)^{i+\sum_{u=1}^{d} c_u}$ and $b'_{i,0} = b'_{i,1-c'_i} = (-1)^{i+\sum_{u=1}^{d} c'_u}$, but since $\sum_{u=1}^{d} c'_u = \sum_{u=1}^{d} c_u + 1$ (because all the corresponding summands are equal, with the exception of $c_i = 0$ and $c'_i = 1$), the two expressions clearly have opposite signs, as needed.

Let us now show that the obtained hypercube behaves like a sample of order $t + 1$. Consider the two facets of the obtained hypercube orthogonal to the i^{th} coordinate axis. The left one is composed of 2^{d-1} left facets (with respect to the same axis) of all the basic hypercubes whose center has 0 at the i^{th} coordinate; by our construction, each of them is marked by a bump (since $b_{i,0} = 1$ whenever the center of the considered hypercube has 0 at the i^{th} coordinate). Consider now the right facet. It is composed of 2^{d-1} facets, and if $2^t < i \leqslant 2^k$, then each of them is marked by a nick, while if $1 \leqslant i \leqslant 2^t$, then each of them is marked by a bump (since for $1 \leqslant i \leqslant 2^t$ we have $b_{i,1} = 1$ whenever the center of the considered hypercube has 1 at the i^{th} coordinate).

Altogether, there are $2^k - 2^t$ facets of the obtained hypercube marked (solely) by nicks, and all the other facets are marked (solely) by bumps; furthermore, there is no i such that both facets orthogonal to the i^{th} coordinate axis are marked by nicks (since all the facets marked by nicks are right ones). Therefore, the obtained hypercube behaves like a sample of order $t+1$ (scaled by factor 2), which was to be proved. ■

We are now ready for the main result.

Theorem 9. *The Heesch number of a basic hypercube in d dimensions, where $d = 2^k$, equals $d - 1$.*

Proof. By Lemma 4, $d - 1$ is the upper bound. For the other direction, as we have already mentioned, we shall prove more: that it is possible to stack $(2d)^d$ basic hypercubes to form a hypercube (up to bumps and nicks) of side $2d$.

By Lemma 6, we can stack 2^d basic hypercubes in such a way to obtain a hypercube that behaves like a sample of order 1 scaled by factor 2. By Lemma 8 for $t = 1$, we can stack 2^d such samples in such a way to obtain a hypercube that behaves like a sample of order 2 scaled by factor 4 (since the initial hypercubes were already scaled by factor 2). By Lemma 8 for $t = 2$, we can further stack 2^d such samples in such a way to obtain a hypercube that behaves like a sample of order 3 scaled by factor 8. Iterating the procedure, after the last step (for $t = k$) we obtain a hypercube that behaves like a sample of order $k+1$ scaled by factor 2^{k+1}. In other words, at the end we get a hypercube (up to bumps and nicks) of side 2^{k+1} (which is $2d$), composed of basic hypercubes, which was to be proved. ■

Note. From the procedure in the above proof it is clear why the constraint $d = 2^k$ is necessary. Namely, the matrix corresponding to a basic hypercube has no rows with two 1's; the matrix corresponding to a sample of order 1 has exactly one row with two 1's, and in general, the matrix corresponding to a sample of order t has exactly 2^{t-1} rows with two 1's. Therefore, after each step in the above proof in which we apply Lemma 8, the number of rows with two 1's doubles, which implies that, if we want to reach the bound from Lemma 4, the dimension (which is the number of rows) has to be a power of 2 (at least for the presented construction).

An open question remains whether our basic hypercubes in other dimensions d (not of the form 2^k) also have Heesch number equal to $d-1$, or (if not) whether a different hypersolid can be constructed that reaches this (or greater) value.

Acknowledgments

The authors would like to thank the two anonymous reviewers for their devoted time and for the careful reading of the article.

The authors were supported by the Ministry of Education, Science and Technological Development of Serbia (grant no. 451-03-68/2020-14/200125).

References

[1] B. Bašić, The Heesch number for multiple prototiles is unbounded, *C. R. Math. Acad. Sci. Paris* **353** (2015), 665–667.

[2] B. Grünbaum & G. C. Shephard, *Tilings and patterns*, W. H. Freeman and Company, New York, 1987.

[3] H. Heesch, Aufbau der Ebene aus kongruenten Bereichen, *Nachr. Ges. Wiss. Göttingen, Neue Folge* **1** (1935), 115–117.

[4] H. Heesch, *Reguläres Parkettierungsproblem*, Westdeutscher Verlag, Cologne, 1968.

[5] C. Mann, *Heesch's Problem and Other Tiling Problems*, PhD thesis, University of Arkansas, 2001.

[6] C. Mann, Heesch's Tiling Problem, *Amer. Math. Monthly* **111** (2004), 509–517.

[7] J. Myers, Polyomino, polyhex and polyiamond tiling, https://www.polyomino.org.uk/mathematics/polyform-tiling/.

[8] K. Reinhardt, Zur Zerlegung der euklidischen Räume in kongruente Polytope, *Sitzungsber. Preuss. Akad. Wiss. Berlin* (1928), 150–155.

[9] J. E. S. Socolar, The hexagonal parquet tiling: k-isohedral monotiles with arbitrarily large k, *Math. Intelligencer* **29** (2007), 33–38.

[10] A. S. Tarasov, On the Heesch number for the Lobachevskiĭ plane, *Mat. Zametki* **88** (2010), 97–104 (Russian); English translation in *Math. Notes* **88** (2010), 97–102.